

Outer space and Automorphisms of free groups

The study of free groups and their automorphisms goes back more than a century, to the 1880's when the concept of "free group" was introduced by Dyck.

Fundamental results were established by J. Nielsen, JHC Whitehead and W. Magnus in the 1920's and 1930's

From the beginning connections with topology and geometry were important, though much of the work was purely algebraic and combinatorial

More recent work on automorphism groups of free groups starts with Stallings, who introduced new topological techniques in the 70's based on fact $\pi_1(\text{graph})$ is free. ("The topology of finite graphs" - Inventiones 1983).

The more recent work is also heavily influenced by ideas of Thurston and Grunewald who used methods from geometry and dynamics to study groups.

Much of this recent work focuses on analogies between $\text{Out}(F_n)$ and mapping class groups of surfaces and lattices in semisimple Lie groups, especially $\text{SL}_n \mathbb{Z}$

One reason to suspect such analogies is that there are natural maps

$$\begin{aligned} \textcircled{1} \quad & \text{Out}(F_n) \longrightarrow \text{GL}(n, \mathbb{Z}) \\ \textcircled{2} \quad & \text{Mod}(S_{g,1s}) \hookrightarrow \text{Out}(F_n) \end{aligned}$$

$$\textcircled{1} \quad \text{Abelianization } F_n \longrightarrow \mathbb{Z}^n \text{ induces } \text{Aut}(F_n) \longrightarrow \text{Aut}(\mathbb{Z}^n) = \text{GL}(n, \mathbb{Z})$$

Since the commutator subgroup $[F_n, F_n]$ is characteristic

Inner automorphisms map to I , so the map factors through $\text{Out}(F_n)$

Nielsen showed $\text{Aut } F_n \rightarrow \text{GL}_n \mathbb{Z}$ is surjective. (1924). (We will see a modern proof)

The kernel of the map $\text{Out}(F_n) \rightarrow \text{GL}(n, \mathbb{Z})$ is called IA_n "Identity on the Abelianization"

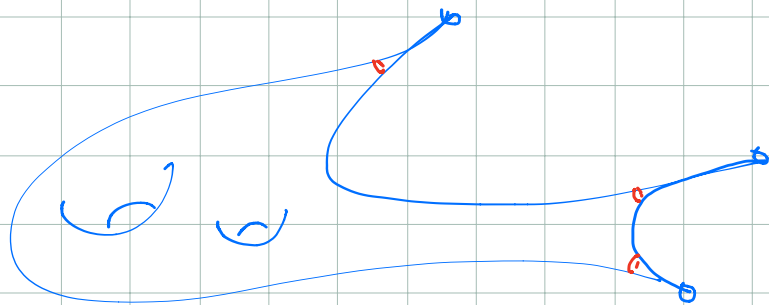
It measures the difference between $\text{Out}(F_n)$ and $\text{GL}(n, \mathbb{Z})$ and is still very poorly understood in general.

Nielsen showed $IA_2 = \langle e \rangle$ in 1924

Magnus showed IA_n is finitely generated and asked whether it is finitely presented in 1934.

In 1997 Krstić & McCool showed IA_3 is not finitely presented, but the question remains open for $n > 3$.

② $S_{g,s}$ = surface with genus g , s punctures



$$\pi_1(S_{g,s}) \cong F_n, \quad n = 2g + s - 1.$$

Any homeomorphism of $S_{g,s}$ induces an automorphism of F_n .

The homeomorphism permutes the punctures, and the (homotopy classes of) simple loops around the punctures, which are cyclic words $u_i \in F_n$.

$$\text{Zieschang} : \text{Mod}(S_{g,s}) \hookrightarrow \text{Out}(\pi_1 S_{g,s}).$$

The image is $\text{Stab}(u_1, \dots, u_s)$, i.e. outer automorphisms which permute the cyclic words u_i .

Remark

There are often many different punctured surfaces w/ $\pi_1 \cong F_n$, many \cong 's.

Stallings: $\exists \varphi \in \text{Out } F_n$ not in the image of any $\text{Mod}(S_{g,s})$.

eg

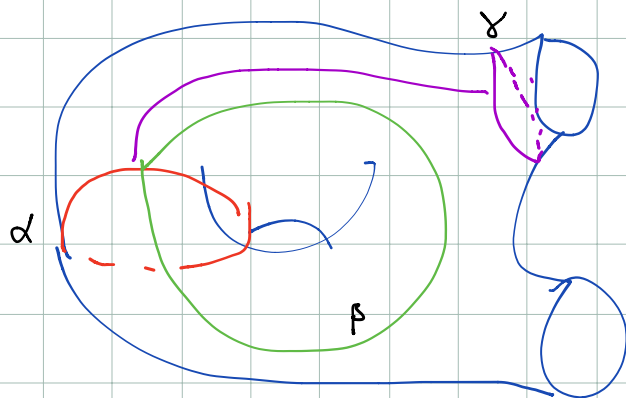
$$\varphi: \begin{cases} x \mapsto y \\ y \mapsto z \\ z \mapsto xy \end{cases}$$

Rank = 3 \Rightarrow could come from a homeomorphism
of a 4-punctured sphere or 2-punctured torus.
Ill do $S = S_{1,2}$ -punctured torus:

If $h: S \rightarrow S$ is a homeomorphism then some
power h^k preserves orientation and fixes ∂S .

Claim $(h^k)_*$: $H_1 S \rightarrow H_1 S$ has eigenvalues
 $1, \lambda, \frac{1}{\lambda}$.

PF:



$\pi_1 S$ generated by α, β, γ
 $H_1 S$ generated by $[\alpha], [\beta], [\gamma]$

$h^k(\gamma) = \gamma$ so $[\gamma]$ is an evector for h^k_*
 \vee eigenvalue 1

So matrix is $\begin{pmatrix} 1 & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \in SL(3, \mathbb{Z})$

\Rightarrow eigenvalues = $\{1, \lambda, \lambda^{-1}\}$

Matrix in blue corner is map on T^2 given by filling in holes and extending h .

Exercise: Complete the proof for $S = S_{0,4}$.

Exercise. For any $h: S_{g,s} \rightarrow S_{g,s}$ preserving orientation and fixing ∂ , eigenvalues of $h_*: H_1 S_{g,s} \rightarrow H_1 S_{g,s}$ include $(s-1)$ 1's, and the rest come in pairs $\{\lambda, \lambda^{-1}\}$

Now back to our automorphism $\varphi: x \rightarrow y \rightarrow z \rightarrow xy$

The induced map on $H_1 S \cong \mathbb{Z}^3$ is $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

which has eigenvalues $\cdot \lambda_1 > 1, \lambda_2, \lambda_3 < 1$.

note $\lambda_1^{-1} = \lambda_2, \lambda_3$ since $\lambda_1 \lambda_2 \lambda_3 = \pm 1$

(Powers of φ have these properties, too)

Lattices are often studied via their action on the symmetric space K/G

Mapping class groups act on the Teichmüller space $\mathcal{T}_{g,s}$

Outer space was introduced in the mid-80's as an analog for $\text{Out}(F_n)$ of symmetric space or Teichmüller space

Outer space can be thought of as a space of graphs.

Many structures in mathematics are parameterized by finite graphs, and so Outer space has found connections, eg, with

- * Moduli spaces of punctured surfaces
- * Systems of spheres in a doubled handlebody
- * tropical curves
- * Feynman diagrams
- * Invariants in symplectic modules
- * Classical modular forms

and even

- * phylogenetic trees

Shortly after Outer space appeared, Bestvina and Handel introduced traintracks for free group automorphisms, as an analog of Thurston's train tracks for surface automorphisms.

The literature has since exploded, and I can't hope to cover all of it in this course. I hope to produce a guide to the literature by the end of the course, for those who want to learn more.

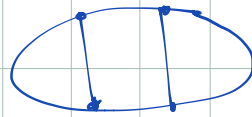
This course will begin at the beginning, and we'll see how far we get.

Start pre-Outer space. Want to study autos of free groups using topology. Need a topological model for a free group; then self-maps of this model can represent automorphisms.

Need $X = \text{topological space}$ w/ $\pi_1 X \cong F_n$

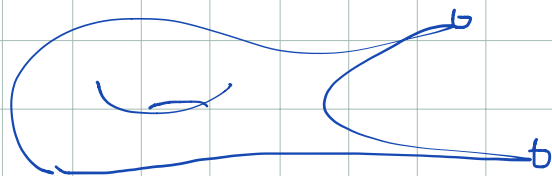
Candidates:

1. Finite connected graph G ($v - e = 1 - n$)

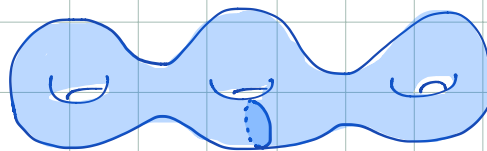


$$v - e = 4 - 6 = -2$$

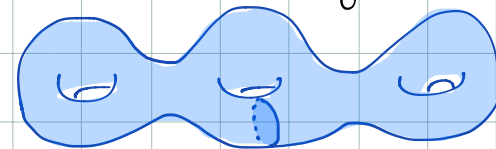
2. Punctured surface $S_{g,s}$ ($2g + s - 1 = n$)



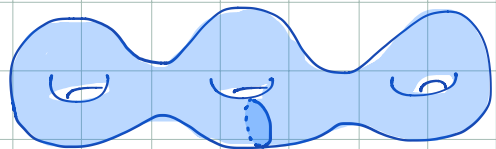
3. Handlebody $H_n = \#_n S^1 \times D^2$



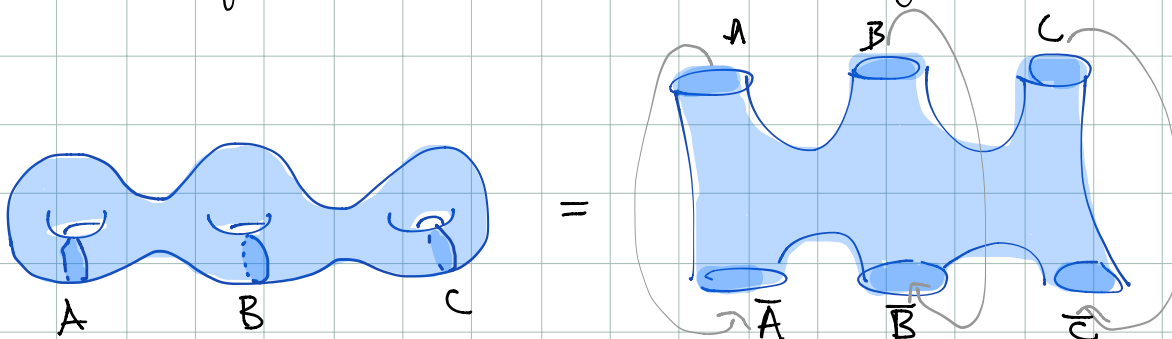
4. Doubled handlebody $M_n = \#_n S^1 \times S^2$



\cup
 \downarrow id on ∂



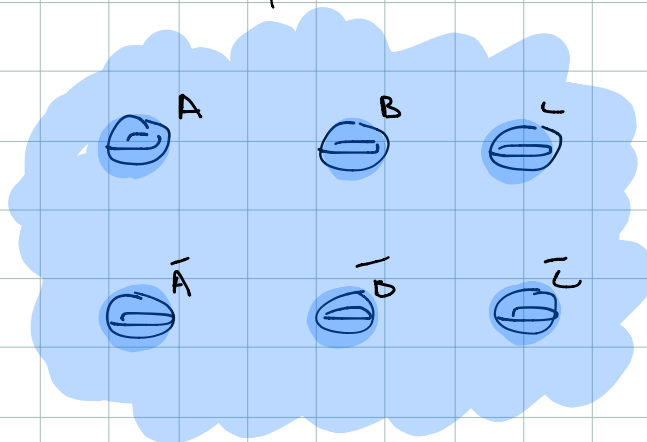
Another way to visualize the doubled handlebody.



After doubling, this becomes

M_n with 3 spheres A, B, C

Sphere S^3 - 3 balls



Any homotopy equivalence $X \rightarrow X$
induces an **outer** automorphism of $\pi_1 X$
(If you want an automorphism, need
to fix a basepoint $b \in X$ and only consider
maps $(X, b) \rightarrow (X, b)$.)

So if you fix an identification $F_n \cong \pi_1 X$
you get an elt of $\text{Out}(F_n)$

Some models are better than others
for studying $\text{Out}(F_n)$

Homotopic maps give the same map on $\pi_1 X$
so get a map $\pi_0 \text{HE}(X) \rightarrow \text{Out}(F_n)$
for any X .

① X a graph: Then this map is an isomorphism

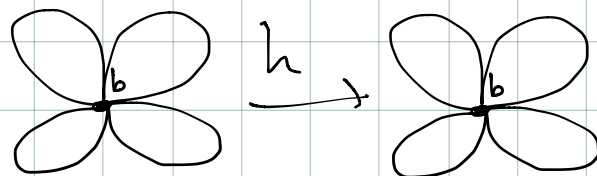
surjective: $X \cong \begin{array}{c} a_2 \quad a_3 \\ \text{---} \text{---} \\ | \quad | \\ a_1 \quad a_4 \end{array} \rightarrow \begin{array}{c} a_2 \quad a_3 \\ \text{---} \text{---} \\ | \quad | \\ a_1 \quad a_4 \end{array} \cong X$

Identify $\pi_1(X) \cong F\langle a_1, \dots, a_n \rangle = F_n$

Given $\varphi \in \text{Aut}(F_n)$, construct a h.equiv.
say $\varphi(a_i) = w_i$ $h: a_i \mapsto w_i$.

Injective: If $h: X \rightarrow X$ induces
 $\text{id}: \pi_1 X \rightarrow \pi_1 X$, then h is homotopic to id .

pf for X a rose:



Composing w a homotopy, if necessary,
 we may assume $h(b) = b$

Each loop a on the left goes to a loop
 homotopic to a . These homotopies can be performed
 independently on each petal, giving a homotopy
 of h to id .

② X a punctured surface $S_{g,s}$

Every h -equiv is homotopic to a homeomorphism

$$S_0 \quad \pi_0(\text{HE}(X)) = \pi_0(\text{Homeo}(X)) = \text{Mod}(X)$$

$$\downarrow \cong \text{Zieschang}$$

$$\text{Stab}(u_1, \dots, u_s) \not\cong \text{Out}(F_n)$$

(except $g=s=1$)

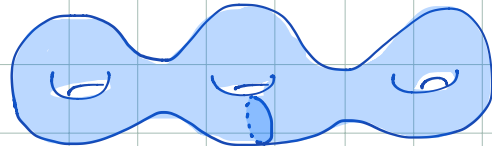
We saw that not every φ can be realized on
 a surface

③ $X = \text{handlebody } H_n$

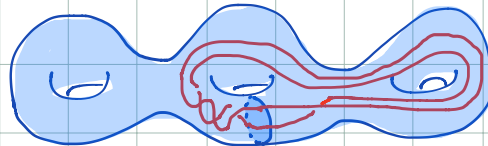
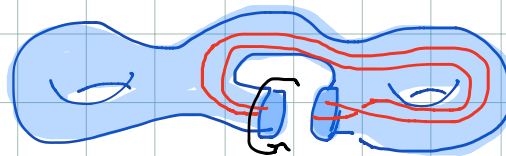
$$\pi_0(\text{Homeo}(H_n)) \longrightarrow \text{Out } F_n$$

is surjective but not injective:

"Dehntwist" on disk



is not $\simeq \text{id}$, but does nothing to $\pi_1 H_n$, so is in kernel.



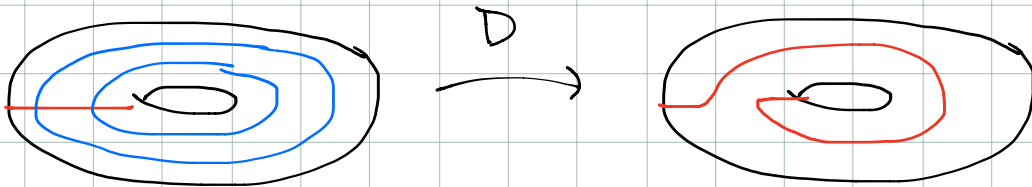
④ $X = \text{doubled handlebody } M_n$

$$\pi_0(\text{Homeo } M_n) \longrightarrow \text{Out}(F_n) \text{ is surjective}$$

Also not injective, but closer:

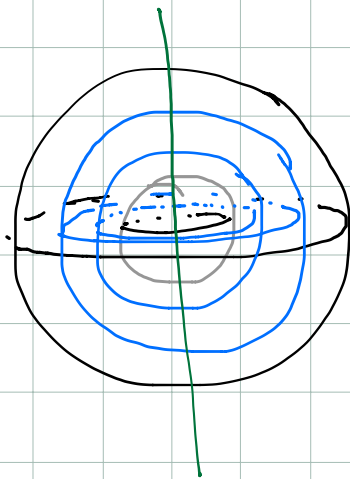
Thm (Laudenbach) kernel is generated by Dehn twists in 2-spheres $S^2 \subset M$

Dehn twist in a surface: supported in an annulus



concentric circles rotate thru angles $0 < t < 2\pi$

In a 3-manifold, Dehn twist supported on $S^2 \times I$



Pick an axis of rotation
Fix outside sphere, rotate
more and more as you go inside,
rotate 360 on inside sphere
(= loop in $SO(3)$).

Get a path in $SO(3)$, in fact a loop.

$$\pi_1(SO(3)) = \mathbb{Z}/2\mathbb{Z} \Rightarrow D^2 = \text{id}.$$

Laudenbach shows kernel is a finite 2-group
 $(\mathbb{Z}/2)^n$

We will use ① and ④ to study automorphisms.

We're not the first:

④ was used by Whitehead, eg to answer
when is $a_i \mapsto w_i$ an automorphism?

eg $\begin{cases} a \mapsto ab \\ b \mapsto b\bar{a}'b\bar{c}' \\ c \mapsto cab \end{cases}$ Is this invertible?

A: No. But $\begin{cases} a \mapsto ab \\ b \mapsto bcab \\ c \mapsto b\bar{a}'c \end{cases}$ is

Whitehead invented a combinatorial algorithm to decide, for any set of distinct cyclic words $\{w_i\}$, whether $a_i \mapsto w_i$ is an automorphism

Nielsen had shown that the automorphisms of $F_n = F(a_1, \dots, a_n)$

$$\rho_{ij} : \begin{cases} a_i \mapsto a_i a_j \\ a_k \mapsto a_k \quad k \neq i \end{cases}$$

$$\lambda_{ij} : \begin{cases} a_i \mapsto a_j a_i \\ a_k \mapsto a_k \quad k \neq i \end{cases}$$

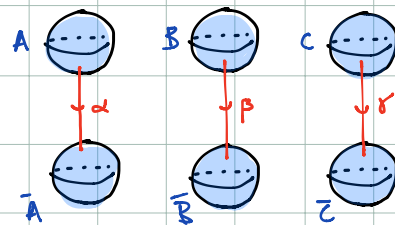
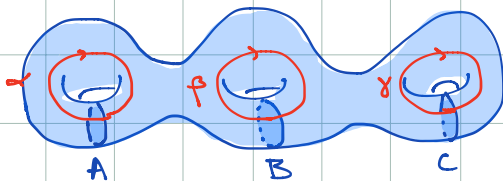
$\varepsilon_i : a_i \mapsto a_i^{-1}$ generate $\text{Aut}(F_n)$ (\therefore hence $\text{Out}(F_n)$)

Assume this for now

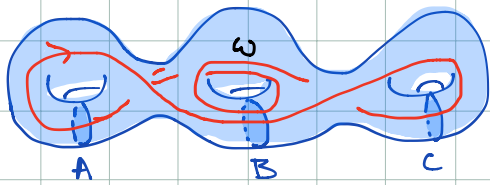
Claim $\pi_0 \text{Diff } M_n \rightarrow \text{Out}(F_n)$ is surjective

By Nielsen, it

suffices to realize $\rho_{ij}, \lambda_{ij}, \varepsilon_i$ on M_n .



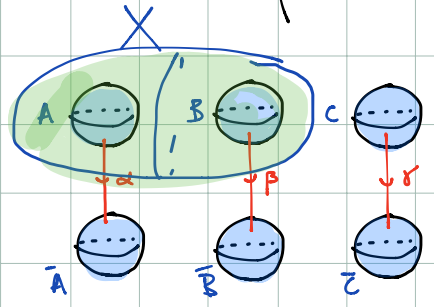
α, β and γ generate $\pi_1(M_n)$ (were ignoring basepts)
 w another loop (=elt of $\pi_1 M_n$)



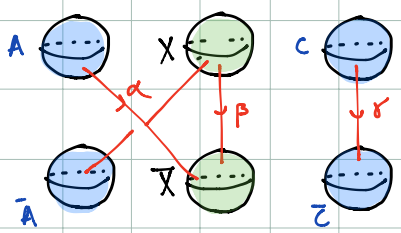
$$w = b \bar{c}' b \bar{a}'$$

you can read off w as a cyclic word in F_3 by looking at which spheres it punctures (in which direction)

To realize $\rho_{ab}: a \mapsto ab$



Choose a diffeomorphism sending $A \rightarrow A, B \leftrightarrow X, C \rightarrow C$



This sends $a \mapsto ab$
 $b \mapsto b$
 $c \mapsto c$

